THE FOURTH DUALS OF BANACH ALGEBRAS

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ABSTRACT. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}^{**}$ the second dual of $\mathcal{A}$ is a Banach algebra with first (second) Arens product. We study the Arens products of $\mathcal{A}^4 (= (\mathcal{A}^{**})^{**})$. We found some conditions on $\mathcal{A}^{**}$ to be a left ideal in $\mathcal{A}^4$. We found the biggest two sided ideal I of $\mathcal{A}$, in which I is a left (right) ideal of $\mathcal{A}^{**}$.

1. Introduction

The regularity of bilinear maps on norm spaces, was introduced by Arens in 1951 [1]. Let X , Y and Z be normed spaced and let $f : X \times Y \longrightarrow Z$ be a continuous bilinear map, then $f^* : Z^* \times X \longrightarrow Y^*$ (the transpose of f) is defined by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \qquad (z^* \in Z^*\ ,\ x \in X\ ,\ y \in Y).$$

(f^* is a continuous bilinear map). Clearly, for each $x \in X$, the mapping $z^* \longmapsto f^*(z^*, x) : Z^* \longrightarrow Y^*$ is $weak^* - weak^*$ continuous. We take $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$,... .

Let X, Y and Z be Banach spaces and let $f : X \times Y \longrightarrow Z$ be a bilinear map. Let X and Z be dual Banach spaces then we define

$$Z_r(f) := \{y \in Y : f(.,y) : X \longrightarrow Z \text{ is } weak^* - weak^* - continuous\} \qquad (1.1),$$

so if Y and Z are dual spaces then we define

$$Z_l(f) := \{x \in X : f(x,.) : Y \longrightarrow Z \text{ is } weak^* - weak^* - continuous\} \qquad (1.2).$$

We call $Z_r(f)$ and $Z_l(f)$, the topological centers of f. For example if $\mathcal{A}$ is a Banach algebra by product $\pi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$, then $\mathcal{A}^{**}$ the second dual of $\mathcal{A}$ is a Banach algebra by each of products π^{***} and π^{r***r} (this products are the first and the second Arens products of $\mathcal{A}^{**}$ respectively). Also we have $Z_l(\pi^{***}) = Z_1$ the left topological center of $\mathcal{A}^{**}$ and $Z_r(\pi^{r***r}) = Z_2$ the right topological center of $\mathcal{A}^{**}$ (see [2], [4], [7]).

Let $Z = \{a'' \in \mathcal{A}^{**} : \pi^{***}(a'', b'') = \pi^{r***r}(a'', b'')$ for every $b'' \in \mathcal{A}^{**}\}$. Then it is easy to show that $\mathcal{A} \subseteq Z$ if and only if $\mathcal{A}$ is commutative.

Lemma 1.1. Let $\mathcal{A}$ be a commutative Banach algebra. Then the following assertions hold.

(i) If $\mathcal{A}^{**}$ has identity E for one of the Arens products then E is identity for other product.

(ii) For every $a'' \in \mathcal{A}^{**}$, we have $\pi^{***}(a'', a'') = \pi^{r***}(a'', a'') = \pi^{r***r}(a'', a'')$.

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Proof. (i) Let E be the identity for $(\mathcal{A}^{**}, \pi^{***})$. Then for every $F \in \mathcal{A}^{**}$, we have

$$\pi^{***r}(F,E) = \pi^{***}(E,F) = F = \pi^{***}(F,E) = \pi^{r***}(F,E) = \pi^{r***r}(E,F).$$

Similarly we can show that E is the identity for $(\mathcal{A}^{**}, \pi^{***})$ when E is the identity of $(\mathcal{A}^{**}, \pi^{r***r})$. The proof of (ii) is trivially. ■

In this paper we study the Arens regularity of $\mathcal{A}^{(4)}$. We fined the conditions on $\mathcal{A}^{**}$ to be a left ideal in $\mathcal{A}^{****}$. Finally we fined the biggest two sided ideal I of $\mathcal{A}$ in which I is a left (right) ideal of $\mathcal{A}^{**}$.

2. Arens products of $\mathcal{A}^{****}$

Let $\mathcal{A}$ be a Banach algebra by product $\pi : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$. The fourth dual $\mathcal{A}^{(4)} = (\mathcal{A}^{**})^{**}$ of $\mathcal{A}$, is a Banach algebra by products $\pi^{******}, \pi^{***r***r}, \pi^{r***r***}$ and $\pi^{r******r}$. Also it is easy to show that

$$Z_r(\pi^{******}) = Z_l(\pi^{***r***r}) = Z_r(\pi^{r***r***}) = Z_l(\pi^{r******r}) = \mathcal{A}^{(4)}.$$

Theorem 2.1. Let $\mathcal{A}$ be a Banach algebra, then the following assertions are equivalent

(i) $Z_l(\pi^{r******r}) = \mathcal{A}^{(4)}$

(ii) $Z_l(\pi^{***r***r}) = \mathcal{A}^{(4)}$

(iii) $Z_r(\pi^{******}) = \mathcal{A}^{(4)}$

(iv) $Z_r(\pi^{r***r***}) = \mathcal{A}^{(4)}$.

Proof. We have

$$Z_l(\pi^{******}) = \mathcal{A}^{(4)} \Longleftrightarrow (\mathcal{A}^{**}, \pi^{***}) \text{ is Arens regular } \Longleftrightarrow Z_r(\pi^{***r***r}) = \mathcal{A}^{(4)} \qquad (2.1),$$

and

$$Z_l(\pi^{r***r***}) = \mathcal{A}^{(4)} \Longleftrightarrow (\mathcal{A}^{**}, \pi^{r***r}) \text{ is Arens regular } \Longleftrightarrow Z_r(\pi^{r******r}) = \mathcal{A}^{(4)} \qquad (2.2).$$

On the other hand $\mathcal{A}$ is Arens regular if $\mathcal{A}^{**}$ is Arens regular. Let one of the conditions (i), (ii), (iii) or (iv) holds, then $\mathcal{A}$ is Arens regular; i.e. $\pi^{r***r} = \pi^{***}$. Then we have

$$Z_l(\pi^{r***r***}) = \mathcal{A}^{(4)} \Longleftrightarrow Z_l(\pi^{******}) = \mathcal{A}^{(4)} \qquad (2.3).$$

(2.1),(2.2) and (2.3) imply that (i),...,(iv) are equivalent. ■

Let $\mathcal{A}$ be a commutative Banach algebra. Then the following assertions are equivalent.

(i) $\mathcal{A}$ is Arens regular

(ii) There is $n \in \mathbb{N}$ such that $\mathcal{A}^{(2n)}$ is Arens regular.

(iii) For every $n \in \mathbb{N}$ $\mathcal{A}^{(2n)}$ is Arens regular.

Theorem 2.2. Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity, then $\widehat{\mathcal{A}^{**}}$ is a left ideal of $\mathcal{A}^{(4)}$ if and only if $\mathcal{A}$ is reflexive.

Proof. Let $(e_\alpha)_{\alpha\in I}$ be a bounded right approximate identity for $\mathcal{A}$ with cluster point $E\in\mathcal{A}^{**}$. E is a right unit element of $\mathcal{A}^{**}$, then $\widehat{E}$ is a right unit element of $\mathcal{A}^{(4)}$. If $\mathcal{A}^{**}$ is a left ideal of $\mathcal{A}^{(4)}$ then $\mathcal{A}^4=\mathcal{A}^4\widehat{E}=\widehat{\mathcal{A}^{**}}$, then $\mathcal{A}$ is reflexive. The converse is trivial. ■

Let $\mathcal{A}$ be a Banach algebra in which $\widehat{\mathcal{A}^{**}}$ be a left ideal of $\mathcal{A}^{(4)}$. Then we can show that $\widehat{\mathcal{A}}$ is a left ideal of $\mathcal{A}^{**}$. In the following we show that the converse of the above statement does not hold.

Example 2.3. Let G be a compact topological group, then $L^1(G)$ is an ideal of $L^1(G)^{**}$. On the other hand $L^1(G)$ is reflexive if and only if G is finite. Then by above lemma, $L^1(G)^{**}$ is not a left ideal of $L^1(G)^{(4)}$ when G is infinite.

Let $\pi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ be the product of Banach algebra $\mathcal{A}$, Dales, Rodriguez and Velasco in [3] found some necessary and sufficient conditions for Arens regularity of both π and π^{r*}. They proved the following theorem that plays a key role in [3]. We will use this theorem to show that π^{r**} is Arens regular if and only if $\mathcal{A}^{**}$ is a left ideal of $\mathcal{A}^{****}$ when π and π^{r*} are Arens regular.

Theorem 2.4. Let $f:X\times Y\longrightarrow Z$ be a continuous bilinear map, then the following assertions are equivalent.

(i) f and f^* are Arens regular.

(ii) $f^{r*r***}=f^{***r*r}$.

(iii) $f^{****}(Z^{***},X^{**})\subseteq\widehat{Y^*}$.

corollary 2.5. Let $\mathcal{A}$ be a Banach algebra with product $\pi:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$. Let π and π^{r*} be Arens regular, then π^{r**} is Arens regular if and only if $\mathcal{A}^{**}$ be a left ideal in $\mathcal{A}^{****}$.

Proof: Let $f:=\pi^{r*r}$, then f and f^{r*} are Arens regular if and only if $f^{****}(Z^{***},X^{**})\subseteq\widehat{Y}^*$, we assume that π^{r*r} be Arens regular then π^{r**} is Arens regular if and only if $\pi^{r*r****}(\mathcal{A}^{****},\mathcal{A}^{**})\subseteq\widehat{(\mathcal{A}^{**})}$.

For $a''''\in\mathcal{A}^{****}$, $a'''\in\mathcal{A}^{***}$ and $a''\in\mathcal{A}^{**}$, we have

$$\langle\pi^{***r*r*}(a'''',a''),a'''\rangle=\langle a'''',\pi^{***r*}(a''',a'')\rangle\qquad(2.4),$$

and

$$\langle\pi^{******}(a'''',\widehat{a''}),a'''\rangle=\langle a'''',\pi^{*****}(\widehat{a''},a''')\rangle\qquad(2.5).$$

Also for every $b''\in\mathcal{A}^{**}$, we have

$$\begin{aligned}\langle\pi^{***r*}(a''',a''),b''\rangle&=\langle a''',\pi^{***r}(a'',b'')\rangle=\langle a''',\pi^{***}(b'',a'')\rangle\\&=\langle\pi^{****}(a''',b''),a''\rangle=\langle\widehat{a''},\pi^{****}(a''',b'')\rangle\\&=\langle\pi^{*****}(\widehat{a''},a'''),b''\rangle\qquad(2.6).\end{aligned}$$

By (2.1), (2.2) and (2.3), we have $\pi^{r*r****}(a'''',a'')=\pi^{******}(a'''',\widehat{a''})$. On the other hand since π and π^{r*} are Arens regular, then by theorem 3.1, we have $\pi^{r*r****}=(\pi^{r*r***})^*=(\pi^{***r*r})^*=\pi^{***r*r*}$ and since π^{******} is the first Arens product of $\mathcal{A}^{****}$, then π^{r**} is Arens regular if and only if $\mathcal{A}^{**}$ is a left ideal in $\mathcal{A}^{****}$. ■

The conditions in above corollary are very strung. If $\mathcal{A}$ has bounded approximate identity, and if π, π^{r*} and π^{r**} are Arens regular, then it is easy to show that $\mathcal{A}$ is reflexive. But Arens regularity of π, π^{r*} and π^{r**} dos not implies the rflexivity of $\mathcal{A}$, always. For example if $\mathcal{A}$ is a nonreflexive Banach space with trivial product, then π, π^{r*} and π^{r**} are Arens regular, but $\mathcal{A}$ is not reflexive.

3. Some new ideals

Let $\mathcal{A}$ be a Banach algebra. We consider

$$Z_l(\mathcal{A}) := \{a \in \mathcal{A} \ : \ \mathcal{A}^{**}.\widehat{a} \subseteq \widehat{\mathcal{A}}\} \qquad (3.1),$$

$$Z_r(\mathcal{A}) := \{a \in \mathcal{A} \ : \ \widehat{a}.\mathcal{A}^{**} \subseteq \widehat{\mathcal{A}}\} \qquad (3.2).$$

In this section we show that for a locally compact group G, G is compact if and only if $Z_l(M(G)) = L^1(G)$ (if and only if $Z_r(M(G)) = L^1(G)$), and G is finite if and only if $Z_l(M(G)) = M(G)$ (if and only if $Z_r(M(G)) = M(G)$). Also we show that for a semigroup S, sS is finite for every $s \in S$ if and only if $Z_r(l^1(S)) = l^1(S)$. So in the case that $l^1(S)$ is semisimple, we fined the necessary and sufficient condition on S on which $l^1(S)$ to be an annihilator algebra.
It is easy to show that $Z_l(\mathcal{A})$ and $Z_r(\mathcal{A})$ are two sided ideals of $\mathcal{A}$ and $Z_l(\mathcal{A})$ ($Z_r(\mathcal{A})$) is a left (right) ideal of $\mathcal{A}^{**}$. Also $Z_l(\mathcal{A})$ ($Z_r(\mathcal{A})$) is the union of all two sided ideals of $\mathcal{A}$ which are left (right) ideals of $\mathcal{A}^{**}$. Therefore we result the following.

Theorem 3.1. Let $\mathcal{A}$ be a Banach algebra. Then the following assertions are equivalent.

(i) $\mathcal{A} = Z_l(\mathcal{A})(Z_r(\mathcal{A}))$.

(ii) $\mathcal{A}$ is a left (right) ideal in $\mathcal{A}^{**}$.

(iii) For every $a \in \mathcal{A}$, the map $b \longmapsto ab(b \longmapsto ba) : \mathcal{A} \longrightarrow \mathcal{A}$ is weakly compact.

Proof. $(i) \Longleftrightarrow (ii)$ is straightforward and for $(ii) \Longleftrightarrow (iii)$ see Proposition 1.4.13 of [9]. ■

Theorem 3.2. Let $\mathcal{A}$ be a Banach algebra. Then

$$Z_l(\mathcal{A}) = \{a \in \mathcal{A} \ : \ \text{the mapping } f \longmapsto af : \mathcal{A}^* \longrightarrow \mathcal{A}^* \text{ is } weak^* - weak - \text{ continuous }\},$$

and

$$Z_r(\mathcal{A}) := \{a \in \mathcal{A} \ : \ \text{the mapping } f \longmapsto fa : \mathcal{A}^* \longrightarrow \mathcal{A}^* \text{ is } weak^* - weak - \text{ continuous }\}.$$

Proof. We set

$$U = \{a \in \mathcal{A} \ : \ \text{the mapping } f \longmapsto af : \mathcal{A}^* \longrightarrow \mathcal{A}^* \text{ is } weak^* - weak - \text{ continuous }\},$$

and let $a \in U$ and $b \in \mathcal{A}$. If $f_\alpha \xrightarrow{weak^*} f$ in $\mathcal{A}^*$, then $bf_\alpha \xrightarrow{weak^*} bf$ in $\mathcal{A}^*$. By definition of U, $abf_\alpha \xrightarrow{weak} abf$ in $\mathcal{A}^*$. Thus $ab \in U$. On the other hand since $a \in U$, then $af_\alpha \xrightarrow{weak} af$ in $\mathcal{A}^*$ and because $\mathcal{A}^*$ is a dual Banach space, then every bounded linear map on $\mathcal{A}^*$ is weak-weak

continuous, therefore we have $baf_\alpha \xrightarrow{weak} baf$ in $\mathcal{A}^*$. Thus $ba \in U$ and U is a two sided deal in $\mathcal{A}$. Let now $a'' \in \mathcal{A}^{**}$, $a \in U$ and $f_\alpha \xrightarrow{weak^*} f$ in $\mathcal{A}^*$. Then $af_\alpha \xrightarrow{weak} af$ in $\mathcal{A}^*$ and we have

$$\lim_\alpha \langle a''\widehat{a}, f_\alpha\rangle = \lim_\alpha \langle a'', af_\alpha\rangle = \langle a'', af\rangle = \langle a''\widehat{a}, f\rangle.$$

Thus $a''\widehat{a} : \mathcal{A}^{**} \longrightarrow \mathbb{C}$ is $weak^*$-$weak^*-$ continuous then $a''\widehat{a} \in \widehat{\mathcal{A}}$ and $U \subseteq Z_l(\mathcal{A})$.
Let now $a'' \in \mathcal{A}^{**}$, $a \in Z_l(\mathcal{A})$ then $a''\widehat{a} \in \widehat{\mathcal{A}}$. Suppose $f_\alpha \xrightarrow{weak^*} f$ in $\mathcal{A}^*$. Then

$$\lim_\alpha \langle a'', af_\alpha\rangle = \lim_\alpha \langle a''\widehat{a}, f_\alpha\rangle = \langle a''\widehat{a}, f\rangle = \langle a'', af\rangle.$$

Therefore $af_\alpha \xrightarrow{weak} af$ in $\mathcal{A}^*$ and $Z_l(\mathcal{A}) \subseteq U$. Similarly we can prove the argument for $Z_l(\mathcal{A})$.■

Example 3.3. Let $\mathcal{A} = l^1(\mathbb{N})$ with product $fg = f(1)g$ $(f, g \in l^1(\mathbb{N}))$. Then $\mathcal{A}$ is a Banach algebra with l^1-norm. $\mathcal{A}$ is a left ideal of $\mathcal{A}^{**}$ and we have $\mathcal{A} = Z_l(\mathcal{A})(Z_r(\mathcal{A}))$. On the other hand $Z_r(\mathcal{A}) = \{f \in \mathcal{A} : f(1) = 0\}$. Thus $Z_l(\mathcal{A})$ and $Z_r(\mathcal{A})$ are different.

Let $\mathcal{A}$ be a Banach algebra and let $\mathcal{A}^{**}$ be a left (right) ideal of $\mathcal{A}^{(4)}$, then $\mathcal{A}$ is a left (right) ideal of $\mathcal{A}^{**}$. Therefore we have the following .

(i) If $Z_l(\mathcal{A}^{**}) = \mathcal{A}^{**}$, then $Z_l(\mathcal{A}) = \mathcal{A}$.

(ii) If $Z_r(\mathcal{A}^{**}) = \mathcal{A}^{**}$, then $Z_r(\mathcal{A}) = \mathcal{A}$.

Theorem 3.4. Let G be a locally compact group. Then $Z_l(L^1(G)) = Z_l(M(G))$ and $Z_r(L^1(G)) = Z_r(M(G))$.

Proof. Because $L^1(G)$ has a bounded approximate identity, then by Cohen factorization theorem, $(L^1(G))^2 = L^1(G)$. On the other hand $L^1(G)$ is a two sided ideal of $M(G)$. Therefore every ideal of $L^1(G)$ is an ideal of $M(G)$. Let $\pi : L^1(G) \longrightarrow M(G)$ be the inclusion map, then $\pi''(l^1(G))^{**}$ is a two sided ideal of $M(G)^{**}$. Thus every left (right or two sided) ideal of $\pi''(l^1(G))^{**}$ is a left (right or two sided) ideal of $M(G)^{**}$. Then $Z_l(L^1(G)) \subseteq Z_l(M(G))$. We have to show that $Z_l(M(G)) \subseteq Z_l(L^1(G))$. To this end let (e_α) be a bounded approximate identity of $L^1(G)$ with bound 1 and with a cluster point $E \in L^1(G)^{**}$. Then the mapping $m \longmapsto (\pi''(E))\widehat{m} : M(G) \longrightarrow \pi''(L^1(G))^{**}$ is isometric embedding. We denote this map with Γ_E. Since the restriction of Γ_E to $L^1(G)$ is identity map, then $\Gamma_E(m) \in \pi(\widehat{L^1(G)})$ if and only if $m \in L^1(G)$. Let now $m \in Z_l(M(G))$ then $M(G)^{**}\widehat{m} \subseteq \widehat{M(G)}$. Thus $\pi''(L^1(G))^{**}\widehat{m} \subseteq \widehat{M(G)}$. Since $\pi''(L^1(G))^{**}$ is an ideal of $M(G)^{**}$, we have $\pi''(L^1(G))^{**}\widehat{m} \subseteq [(\widehat{M(G)}) \cap \pi''(L^1(G))^{**}] = \widehat{L^1(G)}$ (see corollary 3.4 of [6]). Therefore $\Gamma_E(m) \in \pi''(L^1(G))^{**}$ and $m \in L^1(G)$. This conclude that $Z_l(M(G)) \subseteq Z_l(L^1(G))$. Similarly we can show that $Z_r(M(G)) = Z_r(L^1(G))$. ■

Corollary 3.5. For a locally compact group G the following assertions are equivalent.

(i) G is compact.

(ii) $Z_l(L^1(G)) = L^1(G)$ ($Z_r(L^1(G)) = L^1(G)$).

(iii) $Z_l(M(G)) = L^1(G)$ ($Z_r(M(G)) = L^1(G)$).

Theorem 3.6. Let S be a semigroup, then

(i) sS is finite for every $s \in S$.

(ii) $Z_r(l^1(S)) = l^1(S)$.

Proof. (i)$\Longrightarrow$(ii). We have to show that for every $a \in l^1(S)$, $\lambda_a : l^1(S) \longrightarrow l^1(S)$ is compact operator. To this end, let $a \in l^1(S)$, then $a = \sum_{n=1}^{\infty} a_n s_n$ when $c_n = a(s_n)$. Since $s_n S$ is finite for every n, then $\lambda_{s_n}(l^1(S))$ is a finite dimension subspace of $l^1(S)$. Thus λ_{s_n} is compact operator on $l^1(S)$. Therefore the operator $\sum_{n=1}^{k} c_n \lambda_{s_n}$ is compact for every $k \in \mathbb{N}$. But $\lambda_a = \sum_{n=1}^{\infty} a_n \lambda_{s_n}$, then by VI. 5.3. of [5], the set of compact operators is closed in the uniform operator topology of $BL(X, Y)$ and we get λ_a is a compact operator on $l^1(S)$.

(ii)$\Longrightarrow$(i). Let $s_0 \in S$ and $s_0 S$ be infinite. Then there exists $\{u_n\}_{n \in \mathbb{N}} \subseteq S$ such that $s_0 u_n \neq s_0 u_m$ when $n \neq m$. Then λ_{s_0} is isometric on an infinite dimension subspace of $l^1(s)$. i.e. λ_{s_0} is not compact. ■

Corollary 3.7. Let $l^1(S)$ be semisimple. Then the following assertions are equivalent.

(i) $Z_r(l^1(S)) = Z_l(l^1(S)) = l^1(S)$ and $S = \{st : s, t \in S\}$.

(ii) $l^1(S)$ is an annihilator algebra.

Proof. (i)$\Longrightarrow$(ii). Let $s \in S$ and let (i) holds. Then SsS is finite therefore $l^1(S)sl^1(S)$ is finite dimension. Since $l^1(S)$ is semisimple and $l^1(S)sl^1(S)$ is an ideal of $l^1(S)$, then $l^1(S)sl^1(S)$ is semisimple finite dimension ideal of $l^1(S)$. Therefore $l^1(S)sl^1(S)$ is isomorphic with the direct sum of full matrix algebras. Now, let $P \in S$. Then $P = s_1 s_2$ for some s_1 and s_2 in S and we have $s_2 = t_1 t_2$ for $t_1, t_2 \in S$. Thus $P = s_1 t_1 t_2 \in S t_1 S$ for some $t_1 \in S$. On the other hand for each $a \in l^1(S)$ we have $a = \sum_{n=1}^{\infty} C_n P_n$ where $P_n \in S$ and $a(P_n) = C_n$. We get that $l^1(S)$ is the topological sum of full matrix algebras, and by 2.8.29 of [8], $l^1(S)$ is an annihilator algebra. (ii)$\Longrightarrow$(i). Since $l^1(S)$ is semisimple annihilator algebra, then by theorem 3.1. of [10] $\widehat{l^1(S)}$ is a two sided ideal of $(l^1(S))^{**}$. Then by above theorem, Ss and sS are finite for every $s \in S$. To prove $S = \{st : s, t \in S\}$ we have $l^1(S)l^1(S) \subseteq l^1(S^2)$ where $l^1(S^2)$ is a closed two sided ideal of $l^1(S)$. Now $l^1(S)$ is an annihilator algebra, then

$$\begin{aligned}
ran(l^1(S)) = \{0\} \quad &\Longrightarrow \quad ran(l(S))^2 = \{0\} \\
&\Longrightarrow \quad ran(l^1(S^2)) = \{0\} \\
&\Longrightarrow \quad l^1(S^2) = l^1(S) \\
&\Longrightarrow \quad S^2 = S.
\end{aligned}$$

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